Module-3: Pfaffian Differential Equations

1. Definition

Let F_i, $(i = 1, 2, \cdots, n)$, be n functions of some or all of the n independent variables $x_1, x_2, \cdots, x_n$. The the expression of the form

$$\sum_{i=1}^{n} F_i(x_1, x_2, \cdots, x_n)dx_i \tag{1}$$

is called a *Pfaffian differential form* and the equation

$$\sum_{i=1}^{n} F_i(x_1, x_2, \cdots, x_n)dx_i = 0 \tag{2}$$

is known as *Pfaffian differential equation.*

It is to be noted that there is a fundamental difference between Pfaffian differential equations in two variables and those in higher number of variables.

In the case of two variables x, y, we can express the equation (2) in the form

$$P(x,y)dx + Q(x,y)dy = 0, \text{ i.e. } \frac{dy}{dx} = f(x,y) \tag{3}$$

where $f(x,y) = -P/Q$. If P and Q are defined and single-valued in the xy-plane, then $f(x,y)$ is also defined uniquely and is single-valued in the same plane. Thus the solution of the equation (3) subject to the boundary condition $y = y_0$ at $x = x_0$ consists of the curve passing through this point and the tangent at each point of the curve is defined by (2). Hence, the differential equation (3) defines a one-parameter family of curves in the xy-plane, i.e there exists a function of the type

$$\phi(x,y) = c, \tag{4}$$

c being constant, which defines a function $y(x)$ satisfying the differential equation (3) identically at least in a certain region in the xy-plane.

The differential form $Pdx + Qdy$ is said to be exact or integrable if it can be written in the form $d\phi(x,y)$. Otherwise, we write the equation (4) as the differential form

$$\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = 0$$

and, hence, there exists a function $\mu(x,y)$ such that

$$\frac{1}{P}\frac{\partial\phi}{\partial x} = \frac{1}{Q}\frac{\partial\phi}{\partial y} = \mu(x,y).$$

Multiplying both sides of the equation (4) by $\mu(x,y)$, we get

$$\mu(Pdx + Qdy) = d\phi = 0.$$

The function $\mu(x,y)$ is called an *integrating factor* of the Pfaffian differential equation (3). *Thus we have the following theorem:*

Theorem 1: *For two variables, Pfaffian differential equation always possesses an integrating factor.*

Next, consider Pfaffian differential equation in three variables x,y,z of the type

$$P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz = 0 \tag{5}$$

which can be written in vector form as

$$\mathbf{X}\cdot\mathbf{dr} = 0 \tag{6}$$

where $\mathbf{X} = (P,Q,R)$ and $\mathbf{dr} = (dx,dy,dz)$.

Before the discussions of the equations of the type (5) or (6), we prove the following two lemmas:

Lemma 1: *A necessary and sufficient condition for the existence of a relation of the form $F(u,v) = 0$ between two functions $u(x,y)$ and $v(x,y)$, not involving x or y explicitly, is that*

$$\frac{\partial(u,v)}{\partial(x,y)} = 0. \tag{7}$$

Proof: Necessity:

Since

$$F(u,v) = 0. \tag{8}$$

is an identity in x and y, so differentiating this with respect to x and y, we get respectively

$$\frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} = 0 \text{ and } \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y} = 0$$

Eliminating $\frac{\partial F}{\partial v}$ between these equations, we get

$$\frac{\partial F}{\partial u} \frac{\partial(u,v)}{\partial(x,y)} = 0.$$

Since (8) involves both u and v, so $\frac{\partial F}{\partial u} \neq 0$ and hence

$$\frac{\partial(u,v)}{\partial(x,y)} = 0.$$

Sufficiency:

Eliminating y between $u = u(x,y)$ and $v = v(x,y)$, we obtain a relation of the type

$$F(u,v,x) = 0.$$

Differentiating with respect to x and y we have respectively

$$\begin{aligned}\frac{\partial F}{\partial x} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} &= 0,\\ \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y} &= 0.\end{aligned}$$

Elimination of $\frac{\partial F}{\partial v}$ between these equations leads to

$$\frac{\partial F}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial F}{\partial u}\frac{\partial(u,v)}{\partial(x,y)} = 0, \text{ i.e. } \frac{\partial F}{\partial x}\frac{\partial v}{\partial y} = 0. \text{ (using (7))}$$

Since $v = v(x,y)$, so $\frac{\partial v}{\partial y} \neq 0$ and hence $\frac{\partial F}{\partial x} = 0$ i.e. F does not contain x explicitly. Similarly, F does not contain y explicitly.

Lemma 2: *Let* $\mathbf{X}$ *be a vector function of* x, y, z *and* $\mathbf{X} \cdot (\nabla \times \mathbf{X}) = 0$ *and* μ *is a function of* x, y, z, *then* $(\mu\mathbf{X}) \cdot \{\nabla \times (\mu\mathbf{X})\} = 0$.

Proof: Let $\mathbf{X} = (P,Q,R)$. Then

$$\begin{aligned}(\mu\mathbf{X}) \cdot \{\nabla \times (\mu\mathbf{X})\} &= \mu P\left\{\frac{\partial}{\partial y}(\mu R) - \frac{\partial}{\partial z}(\mu Q)\right\} + \mu Q\left\{\frac{\partial}{\partial z}(\mu P) - \frac{\partial}{\partial x}(\mu R)\right\}\\ &\quad + \mu R\left\{\frac{\partial}{\partial x}(\mu Q) - \frac{\partial}{\partial y}(\mu P)\right\}\\ &= \mu^2\left\{P\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + Q\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + R\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\right\}\\ &\quad + \mu\left(PR\frac{\partial \mu}{\partial y} - PQ\frac{\partial \mu}{\partial z} + PQ\frac{\partial \mu}{\partial z} - QR\frac{\partial \mu}{\partial x} + QR\frac{\partial \mu}{\partial x} - PR\frac{\partial \mu}{\partial y}\right)\\ &= \mu^2 \mathbf{X} \cdot (\nabla \times \mathbf{X}) = 0.\end{aligned}$$

Using the factor $\frac{1}{\mu}$, the converse of Lemma 2 follows easily.

We now return to Pfaffian differential equation (5). All such equations may not be integrable. However, if we can find a function $\mu(x,y,z)$ such that the expression $\mu(pdx+Qdy+Rdz)$ is an exact differential $d\phi(x,y,z)$, say, then the equation (5) becomes integrable. The function $\mu(x,y,z)$ is called *integrating factor* and the function $\phi(x,y,z)$ is termed as the *primitive* of the differential equation (5).

Theorem 2: *A necessary and sufficient condition for the Pfaffian differential equation* $\mathbf{X}\cdot\mathbf{dr} = 0$ *to be integrable is that* $\mathbf{X}\cdot(\nabla\times\mathbf{X}) = 0$, *where* $\mathbf{X} = (P,Q,R)$ *and* $\mathbf{dr} = (dx,dy,dz)$.

Proof: Necessity:

Since the equation $\mathbf{X}\cdot\mathbf{dr} = 0$, i.e. $Pdx+Qdy+Rdz = 0$ is integrable, so there exists a relation of the type $\phi(x,y,z) = \text{constant}$, so that

$$\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz = 0$$

and, hence, an integrating factor $\mu(x,y,z)$ exists such that

$$\mu P = \frac{\partial\phi}{\partial x},\;\; \mu Q = \frac{\partial\phi}{\partial y},\;\; \mu R = \frac{\partial\phi}{\partial z},\;\; \text{i.e.}\;\; \mu\mathbf{X} = \nabla\phi.$$

It then follows that $\nabla\times(\mu\mathbf{X}) = \nabla\times(\nabla\phi) = \mathbf{0}$. Thus using Lemma 2, we have $\mathbf{X}\cdot(\nabla\times\mathbf{X}) = 0$.

Sufficiency:

Suppose z is constant. Then the differential equation $\mathbf{X}\cdot\mathbf{dr} = 0$ reduces to $P(x,y,z)dx+Q(x,y,z)dy = 0$ which by Theorem 2 possesses a solution of the form $\phi(x,y,z) = \text{constant} = c_1$, say, which may involve z. Also, there exists a function $\mu(x,y,z)$ such that $\mu P = \frac{\partial\phi}{\partial x}$, $\mu Q = \frac{\partial\phi}{\partial y}$.

Then the equation $Pdx+Qdy+Rdz = 0$ gives

$$\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\left(\mu R-\frac{\partial\phi}{\partial z}\right)dz = 0,\;\; \text{i.e.}\; d\phi+\psi dz = 0 \tag{9}$$

where $$d\phi = \frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy \;\;\text{and}\;\; \psi = \mu R-\frac{\partial\phi}{\partial z}. \tag{10}$$

Since $$\mu\mathbf{X} = (\mu P,\; \mu Q,\; \mu R) = \left(\frac{\partial\phi}{\partial x},\; \frac{\partial\phi}{\partial y},\; \frac{\partial\phi}{\partial z}+\psi\right) = \nabla\phi+(0,0,\psi),$$

so $$\mu\mathbf{X}\cdot\{\nabla\times(\mu\mathbf{X})\} = \left(\frac{\partial\phi}{\partial x},\; \frac{\partial\phi}{\partial y},\; \frac{\partial\phi}{\partial z}+\psi\right)\cdot\left(\frac{\partial\psi}{\partial y},\; -\frac{\partial\psi}{\partial x}, 0\right) = \frac{\partial(\phi,\psi)}{\partial(x,y)}.$$

Now, according to Lemma 2, $\mathbf{X}\cdot(\nabla\times\mathbf{X}) = 0$ implies $\mu\mathbf{X}\cdot\{\nabla\times(\mu\mathbf{X})\} = 0$ and, therefore, $\frac{\partial(\phi,\,\psi)}{\partial(x,y)} = 0$. Also, by Lemma 1, there exists a relation between ϕ and ψ which is independent of x and y, but not necessarily of z. In other words, ψ is a function of ϕ and z

only, i.e. $\psi = \psi(\phi, z)$. Then the equation (9) gives $\frac{\partial \phi}{\partial z} + \psi(\phi, z) = 0$ which by Theorem-1 has a solution of the form $\Phi(\phi, z) =$ constant $= c$, say. Replacing ϕ by its expression in terms of x, y, z, we obtain a solution in the form $F(x, y, z) = c$. Hence the equation $\mathbf{X} \cdot \mathbf{dr} = 0$ is integrable.

Theorem 3: *If the differential equation $Pdx + Qdy + Rdz = 0$ has an integrating factor, then one can find an infinity of them.*

Proof: Let $\mu(x, y, z)$ be an integrating factor of the given equation. Then there exists a function $\phi(x, y, z)$ such that

$$\mu P = \frac{\partial \phi}{\partial x}, \quad \mu Q = \frac{\partial \phi}{\partial y}, \quad \mu R = \frac{\partial \phi}{\partial z}.$$

Now, if $\Phi(\phi)$ is an arbitrary function of ϕ, then we can write

$$\begin{aligned} \mu \frac{d\Phi}{d\phi}(Pdx + Qdy + Rdz) = 0 \quad &\Rightarrow \quad \frac{d\Phi}{d\phi}\left(\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz\right) = 0 \\ &\Rightarrow \quad d\Phi = 0 \\ &\therefore \quad \Phi(\phi) = \text{ constant.} \end{aligned}$$

Thus, if μ is an integrating factor yielding a solution $\phi =$ constant and since Φ is an arbitrary function of ϕ, so there exists an infinitely many integrating factors.

Example 1: Verify that the equation $(yz + z^2)dx - zxdy + xydz = 0$ is integrable and find its primitive.

Solution: We have $\mathbf{X} = (yz + z^2, \ -zx, xy)$ so that $\nabla \times \mathbf{X} = (2x, 2z, -2z)$.
Then $\mathbf{X} \cdot (\nabla \times \mathbf{X}) = (yz + z^2)(2x) + (-zx)(2z) + (xy)(-2z) = 0$. Hence the given equation is integrable.

Now, if we take x to be constant, then the given equation reduces to

$$-zxdy + xydz = 0 \quad \Rightarrow \frac{dy}{y} = \frac{dz}{z} \quad \Rightarrow y = c_1 z.$$

$$\text{Thus } \phi(x, y, z) = \frac{y}{z} = c_1 \text{ and so } \mu = \frac{1}{Q}\frac{\partial \phi}{\partial y} = -\frac{1}{zx} \cdot \frac{1}{z} = -\frac{1}{z^2 x}$$

$$\text{and } \psi = \mu P - \frac{\partial \phi}{\partial x} = -\frac{1}{z^2 x}(yz + z^2) = -\frac{y + z}{zx}$$

Hence the equation $d\phi + \psi dx = 0$ leads to

$$\begin{aligned} &\frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz + \psi dx = 0 \\ \text{or} \quad &\frac{1}{z}dy - \frac{y}{z^2}dz - \frac{y + z}{zx}dx = 0 \\ \text{or} \quad &zxd(y + z) = (y + z)d(zx) \end{aligned}$$

Integrating, we have $(y+z) = czx$, c being constant. Thus the required primitive is $y+z = czx$.

2. Solution of Pfaffian differential equations in three Variables

We now discuss some methods of solving Pfaffian differential equations in three variables x, y, z. We assume that the condition of integrability is satisfied.

I. Method-1: (*By inspection*)

In some cases, the differential equation can be solved by inspection. In particular, for the equations when $\nabla \times \mathbf{X} = \mathbf{0}$, then $\mathbf{X} = \nabla\phi$ and the equation $\mathbf{X}\cdot d\mathbf{r} = 0$ i.e. $\nabla\phi \cdot d\mathbf{r} = 0$ gives

$$\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = 0, \text{ i.e. } d\phi = 0$$

whence the primitive is $\phi(x,y,z) = \text{constant}$.

Example 2: Verify that the differential equation $(y^2+yz)dx+(z^2+zx)dy+(y^2-xy)dz = 0$ is integrable and find its primitive.

Solution: Here $\mathbf{X} = (y^2+yz,\ z^2+zx,\ y^2-xy)$ so that $\nabla\times\mathbf{X} = 2(y-z-x, y, -y)$ and hence $\mathbf{X}\cdot(\nabla\times\mathbf{X}) = 2\{(y^2+yz)(y-z-x)+(z^2+zx)y-(y^2-xy)y\} = 0$.
Thus the given equation is integrable.

Now the equation can be written as

$$y(y+z)dx + z(z+x)dy + y(y-x)dz = 0,$$

$$\text{or} \quad y(y+z)d(z+x) + (y+z)(z+x)dy - y(z+x)d(y+z) = 0$$

Dividing both sides by $y(y+z)(z+x)$, we have

$$\frac{d(z+x)}{(z+x)} + \frac{dy}{y} - \frac{d(y+z)}{(y+z)} = 0$$

Integrating, the complete primitive is given by $y(z+x) = c(y+z)$, where c is integration constant.

Example 3: Is the equation $yz(y+z)dx+zx(x+z)dy+xy(x+y)dz = 0$ integrable? Justify your answer. If it is integrable, find the primitive.

Solution: We have $\mathbf{X} = \{yz(y+z),\ zx(x+z),\ xy(x+y)\}$ so that $\nabla\times\mathbf{X} = (0,0,0)$ and hence $\mathbf{X}\cdot(\nabla\times\mathbf{X}) = 0$. Thus the given equation is integrable.

Now the given equation can be written as $yz(x+y+z)dx + zx(x+y+z)dy + xy(x+y+z)dz - xyz(dx+dy+dz) = 0$.
Dividing both sides by $xyz(x+y+z)$ we get

$$\frac{dx}{x} + \frac{dy}{y} - \frac{dz}{z} - \frac{d(x+y+z)}{x+y+z} = 0 \quad \Rightarrow \quad d\{\log\frac{xyz}{x+y+z}\} = 0.$$

Integrating, we have $\log\left(\frac{xyz}{x+y+z}\right)$ = constant, i.e $xyz = c(x+y+z)$ which is the primitive of the given equation, c being integration constant.

II. Method-2: (*Separation of variables*)

If it is possible to write the given equation in the form $P(x)dx + Q(y)dy + R(z)dz = 0$, then the integral surface is obtained as

$$\int P(x)dx + \int Q(y)dy + \int R(z)dz = c,$$

c being the integration constant.

Example 4: Verify that the differential equation $yzdx + 2zxdy - 3xydz = 0$ is integrable and find its complete primitive.

Solution: Here $\mathbf{X} = (yz, 2zx, -3xy)$ so that $\nabla\times\mathbf{X} = (-5x, 4y, z)$ and hence $\mathbf{X}\cdot(\nabla\times\mathbf{X}) = yz(-5x) + 2zx(4y) - 3xy(z) = 0$. Thus the given equation is integrable.

Now dividing both sides of the given equation by xyz, we have

$$\frac{dx}{x} + 2\frac{dy}{y} - 3\frac{dz}{z} = 0$$

which, on integration, leads to the primitive as $xy^2 = cz^3$, c being integration constant.

III. Method-3: (*one variable separable*)

If one of the variable, say x, is separable, then the Pfaffian differential equation can be written as

$$P(x)dx + Q(y,z)dy + R(y,z)dz = 0.$$

Noting that $\mathbf{X} = (P(x), Q(y,z), R(y,z))$ and $\nabla \times \mathbf{X} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial x}, 0, 0\right)$ it follows that the condition of integrability $\mathbf{X} \cdot (\nabla \times \mathbf{X}) = 0$ implies that $\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial x}$. Hence, the expression $Qdy + Rdz$ is an exact differential $\psi(y,z)$, say. Thus, we can write the above differential equation as

$$P(x)dx + d\psi(y,z) = 0 \quad \Rightarrow \quad \int P(x)dx + \psi(y,z) = c$$

Example 5: Verify that the differential equation $(y^2 - z^2)(x^2 - 1)dx - 2zxdy + 2xydz = 0$ is integrable and hence solve it.

Solution: We can write the given equation in the form

$$\frac{x^2 - 1}{x}dx - \frac{2z}{y^2 - z^2}dy + \frac{2y}{y^2 - z^2}dz = 0$$

so that $\mathbf{X} = (\frac{x^2-1}{x}, \frac{-2z}{y^2-z^2}, \frac{2y}{y^2-z^2})$ and $\nabla \times \mathbf{X} = (0,0,0)$, so that the condition of integrability $\mathbf{X} \cdot (\nabla \times \mathbf{X}) = 0$ is satisfied. Thus the equation is integrable.

Now let us rewrite the give equation in the form

$$\left(x - \frac{1}{x}\right)dx - 2\frac{d(y/z)}{(y/z)^2 - 1} = 0$$

Integrating, we have $\frac{1}{2}x^2 - \log x - \log \frac{(y/z)-1}{(y/z)+1} = \log c \quad \Rightarrow \quad y - z = c(y+z)e^{\frac{1}{2}x^2}$, where c is constant.

IV. Method-4: (*Homogeneous equation*)

If the functions $P(x,y,z)$, $Q(x,y,z)$ and $R(x,y,z)$ of the Pfaffian differential equation be homogeneous in x, y and z of the same degree n, say, then this equation can be transformed by the substitutions $y = ux$, $z = vx$, where u and v are function of x only, to the equation of the form

$$\begin{aligned} P(1,u,v)dx + Q(1,u,v)(udx + xdu) + R(1,u,v)(xdv + vdx) &= 0 \\ \text{or} \quad \frac{dx}{x} + A(u,v)du + B(u,v)dv &= 0 \end{aligned} \tag{11}$$

where

$$\begin{aligned} A(u,v) &= \frac{Q(1,u,v)}{P(1,u,v) + uQ(1,u,v) + vR(1,u,v)}, \\ B(u,v) &= \frac{R(1,u,v)}{P(1,u,v) + uQ(1,u,v) + vR(1,u,v)} \end{aligned}$$

The equation (11) can be solved by Method-3.

It may be noted that such type of equation can also be solved if $F(x,y,z) = xP+yQ+zR \neq 0$ and its reciprocal is an integrating factor.

Example 6: Solve the equation $(x^2z-y^3)dx+3xy^2dy+x^3dz = 0$ by verifying the condition of integrability.

Solution: Here $\mathbf{X} = (x^2z-y^3,\ 3xy^2, x^3)$ and so $\nabla\times\mathbf{X} = (0,\ -2x^2,\ 6y^2)$. Thus $\mathbf{X}\cdot(\nabla\times\mathbf{X}) = (x^2z-y^3)\cdot 0+3xy^2(-2x^2)+x^3\cdot 6y^2 = 0$ and hence the given equation is integrable.

Now the functions $x^2z-y^3,\ 3xy^2, x^3$ are homogenous functions of x,y,z of degree 3 and so we put $y = ux$, $z = vx$ to transform the given equation to the form

$$\begin{aligned}
&(vx^3-u^3x^3)dx+3x\cdot u^2x^2(udx+xdu)+x^3(vdx+xdv) = 0\\
\text{or}\quad &\frac{2dx}{x}+\frac{3u^2du+dv}{u^3+v} = 0\\
\text{Integrating,}\quad &x^2(u^3+v) = \text{ constant } = c\\
\text{or}\quad &x^2\left(\frac{y^3}{x^3}+\frac{z}{x}\right) = c,\ \text{ i.e. } y^3+zx^2 = cx
\end{aligned}$$

which is the required solution.

Alternatively, we may solve the equation as follows :

We have $F(x,y,z) = xP+yQ+zR = x(x^2z-y^3)+3xy^3+x^3z = 2(x^3z+xy^3)\neq 0$ and $dF = 2(3x^2zdx+x^3dz+y^3dx+3xy^2dy) = 2(3x^2z+y^3)dx+6xy^2dy+2x^3dz$. The integrating factor is $\mu(x,y,z) = \frac{1}{F} = \frac{1}{2(x^3z+xy^3)}$.

Multiplying both sides of the given equation by $\mu(x,y,z)$, we have

$$\begin{aligned}
&\frac{(x^2z-y^3)dx+3xy^2dy+x^3dz}{2(x^3z+xy^3)} = 0\\
\text{or}\quad &\frac{(3x^2z+y^3)dx+3xy^2dy+x^3dz-2(x^2z+y^3)dx}{2(x^3z+xy^3)}\\
\text{or}\quad &\frac{(3x^2z+y^3)dx+3xy^2dy+x^3dz}{2(x^3z+xy^3)}-\frac{dx}{x} = 0\\
\text{or}\quad &\frac{(dF/2)}{F} = \frac{dx}{x}
\end{aligned}$$

Integrating we get $F = cx^2$, c being a constant. Hence, the required solution is $x^3z+xy^3 = c'x^2$, i.e. $x^2z+y^3 = c'x$, where $c' = c/2$.

V. Method-5: (*Method of reduction*)

If one of the independent variables, say z, is supposed to be constant, then the Pfaffian differential equation reduces to $Pdx+Qdy=0$ whose solution can be obtained in the form $\phi(x,y)=c$, where c is independent of x and y but may depend on z. So, taking differential of $\phi(x,y)$ and then equating this expression with $Pdx+Qdy+Rdz$, we determine c. Integrating $\frac{\partial c}{\partial z}$, the value of c is obtained as a function of z and then $\phi(x,y)=c$ gives the required solution.

Example 7: Verifying the integrability of the differential equation $x(y^2-a^2)dx+y(x^2-z^2)dy-z(y^2-a^2)dz=0$, solve it.

Solution: Here, we take y to be constant and then the given equation reduces to
$x(y^2-a^2)dx-z(y^2-a^2)dz=0$ i.e $xdx-zdz=0$.
Integrating, we have $\phi(x,z)=x^2-z^2=c$, where c is independent of x and z, but may depend on y. Differentials of $\phi(x,z)=c$ give

$$xdx-\frac{1}{2}\frac{\partial c}{\partial y}dy-zdz=0$$

$$\text{or}\quad x(y^2-a^2)dx-\frac{1}{2}(y^2-a^2)\frac{\partial c}{\partial y}dy-z(y^2-a^2)dz=0$$

Comparing this with the given equation, it follows that

$$-\frac{1}{2}(y^2-a^2)\frac{\partial c}{\partial y}=y(x^2-z^2)=cy$$

$$\text{or}\quad \frac{\partial c}{\partial y}+\frac{2cy}{y^2-a^2}=0$$

Integrating, $c(y^2-a^2)=c'$, where c' is an absolute constant. Hence the required solution is $x^2-z^2=\frac{c'}{y^2-a^2}$ i.e. $(x^2-z^2)(y^2-a^2)=c'$.

VI. Method-6: (*Auxiliary equations*)

The condition of integrability of the equation $\mathbf{X}\cdot\mathbf{dr}=0$, given by $\mathbf{X}\cdot(\nabla\times\mathbf{X})=0$, where $\mathbf{X}=(P,Q,R)$, $\mathbf{dr}=(dx,dy,dz)$, can be written as

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0$$

Comparing this with the equation $\mathbf{X}\cdot\mathbf{dr}=0$, i.e. with $Pdx+Qdy+Rdz=0$, we get

$$\frac{dx}{\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}}=\frac{dy}{\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}}=\frac{dz}{\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}}$$

These equations are called *auxiliary equations* which can be solved by the methods discussed earlier.

Suppose $f(x,y,z) = a$ and $g(x,y,z) = b$ be two integrals of the above equations, where a and b are constants. We have to find A and B such that $Adf + Bdg = 0$ becomes identical with the given equation. Then using $f = a$ and $g = b$, the values of A and B can be found out and the required solution is then obtained on integration.

Example 8: Solve the equation $z(z+y^2)dx + z(z+x^2)dy - xy(x+y)dz = 0$ by verifying the condition of integrability.

Solution: Since $\mathbf{X} = \{z(z+y^2),\ z(z+x^2),\ -xy(x+y)\}$, $\nabla \times \mathbf{X} = \{-2(x^2+xy+z),\ 2(y^2+xy+z),\ 2z(x-y)\}$, so $\mathbf{X}\cdot(\nabla\times\mathbf{X}) = -2z(z+y^2)(x^2+xy+z) + 2z(z+x^2) - 2zxy(x+y)(x-y) = 0$ and thus the condition of integrability of the given equation is satisfied and hence the equation is integrable.

Now the auxiliary equations are

$$\frac{dx}{z+x^2+xy} = \frac{dy}{-z-y^2-xy} = \frac{dz}{(y-x)z}.$$

From these equations, it follows that

$$\frac{d(x+y)}{x+y} + \frac{dz}{z} = 0 \quad \text{and} \quad ydx + xdy - dz = 0,$$

leading to the solutions $f(x,y,z) = (x+y)z = \text{constant}$ and $xy - z = \text{constant}$. If the given equation is identical with the equation $Adf + Bdg = 0$, i.e. with $A\{zdx + zdy + (x+y)dz\} + B(ydx + xdy - dz) = 0$, then $A = z - xy$, $B = (x+y)z$. Hence the equation $Adf + Bdg = 0$ gives

$$(z-xy)d\{(x+y)z)\} = (x+y)z\, d(xy-z)$$

leading to the required solution as $(x+y)z = c(xy-z)$.